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Boundary States for D-branes in AdS_3

Arvind Rajaraman* **and Moshe Rozali**†

Serin Laboratory, Rutgers University
Piscataway, NJ 08854, USA

Abstract

We construct boundary states representing D-strings in AdS_3. These wrap twisted conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$, and the boundary states are therefore based on continuous representations only. We check Cardy's condition and find a consistent open string spectrum. The open string spectrum on all the D-branes is the same.

1 Introduction

Perturbative string theory on AdS_3, which is isomorphic to the $\mathrm{SL}(2,\mathbb{R})$ group manifold, is considerably more complicated than string theory on compact group manifolds. (For previous work on $\mathrm{SL}(2,\mathbb{R})$, see [1, 2, 3, 4, 5, 6, 7, 8].) Some of the subtleties of the closed string spectrum were worked out in [9], where a proposal for the closed string spectrum was proposed, and checked by an explicit computation of the partition function.

In this paper, we perform an analogous analysis for open strings in AdS_3 i.e. strings ending on D-branes in AdS_3. D-branes were discussed from a semiclassical point of view in [10, 11, 12]. We will present here a conformal field theory description of these branes as boundary states.

The problem we have to face in constructing boundary states is that their interactions are divergent. This is because unitary representations of $\mathrm{SL}(2,\mathbb{R})$ are all infinite dimensional. Characters of these representations tend to be ill defined.

On the other hand, the simple reason why D-brane interactions are ill-defined is because all the branes we consider have an infinite volume. We show that in an appropriate system of calculation, all divergences can indeed be understood as volume divergences, and as such can be easily regularized.

Once the overlap of these boundary states is regularized, we can then check Cardy's condition. In particular, we can compare the open string spectrum obtained by Cardy's condition to the open string spectrum obtained in [13] by direct quantization of the open string (see also [14]). We find exact agreement, which is strong sup-

*arvindra@muon.rutgers.edu
†rozali@muon.rutgers.edu

port for our approach. Furthermore, our method works generally for all the branes. and the open string spectrum obtained is always consistent. (In fact, we find the open string spectrum to be the same on all the branes.) We believe this is convincing evidence that our boundary states are exact.

We start by reviewing the closed string spectrum of [9]. We also quantize strings winding around the closed timelike curve of $SL(2, \mathbb{R})$. We then review the D-branes, which were found by [10, 11, 12] to wrap conjugacy classes of $SL(2, \mathbb{R})$.

We then use this geometric description to construct the boundary state in the large k limit. This is done by requiring that the overlap of D-branes with closed string modes be restricted to the conjugacy class. This allows us to write an expression for the boundary state which is valid in the large k limit, and more importantly, where the overlaps of boundary states can be calculated. This expression is given at the end of section 3.

Next, we compute the overlaps of these branes and check that Cardy's condition is satisfied. This is a straightforward, though technical, calculation. Finally, we close with a discussion of $\frac{1}{k}$ corrections to the boundary state, where we argue that the only effect of corrections is to renormalize the overall constant in front of the boundary state.

A few comments are in order. First, our brane couples only to states in the continuous representations. This is in contrast to the recent paper [15]. Secondly, only the unflowed representations appear in the boundary state. Thirdly, the construction of the D-brane has many similar aspects to the construction of D-branes in Liouville theory [16]. It would be interesting to explore this further.

2 Closed Strings on AdS_3

2.1 Point Particles on AdS_3

We review here the quantization of a point particle moving in an AdS_3 background. This serves to fix our notations, and introduce some formulae to be used later on.

In cylindrical (global) coordinates, the metric of AdS_3 is

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\phi^2 \tag{1}$$

We will often find it useful to replace the global coordinate system (ρ, ϕ, t) by the coordinates (ψ, χ, t), with:

$$\sinh \psi = \sinh \rho \sin \phi \qquad \cosh \psi \sinh \chi = -\sinh \rho \cos \phi \tag{2}$$

The AdS_3 metric in these coordinates is

$$ds^2 = d\psi^2 + \cosh^2 \psi (-\cosh^2 \chi dt^2 + d\chi^2) \tag{3}$$

The coordinate ψ will be useful in describing branes wrapping twisted conjugacy classes, as they are located at constant values of ψ.

All modes of particles propagating in AdS_3 should fall into representations of the isometry group, namely $\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R$. In general, eigenfunctions of the Laplacian on a group manifold can be written as:

$$D^j_{mm'}(x) = \langle jm|g(x)|jm'\rangle \tag{4}$$

where $|jm\rangle$ is a basis for a representation of the group, and $g(x)$ is some parametrization of the group manifold. To make those functions explicit in our case we write expressions for the generators of $\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R$, as discussed by [17].

In global coordinates these generators are (see the Appendix for details)

$$\begin{aligned} J^3 &= \frac{i}{2}\partial_u \\ J^+ &= \frac{i}{2}e^{-2iu}\left[\coth 2\rho\partial_u - \frac{1}{\sinh 2\rho}\partial_v + i\partial_\rho\right] \\ J^- &= \frac{i}{2}e^{2iu}\left[\coth 2\rho\partial_u - \frac{1}{\sinh 2\rho}\partial_v - i\partial_\rho\right] \end{aligned} \tag{5}$$

where

$$u = \frac{1}{2}(t+\phi) \qquad\qquad v = \frac{1}{2}(t-\phi) \tag{6}$$

The generators of the other $\mathrm{SL}(2,\mathbb{R})$ algebra are obtained by exchanging u and v in the above expressions.

We can now discuss the mode expansion for a massive scalar field, of mass M, in AdS_3. The discussion follows the notation in [17].

The eigenvalue equation for the massive scalar field is:

$$\Box\Phi = \partial_\rho^2 + 2\frac{\cosh 2\rho}{\sinh 2\rho}\partial_\rho + \frac{1}{\sinh^2\rho}\partial_\phi^2 - \frac{1}{\cosh^2\rho}\partial_t^2 = M^2\Phi \tag{7}$$

The eigenvalues of the Laplacian are parametrized by j, which is defined through $M^2 = 4j(j-1)$.

The general mode can be characterized by 3 quantum numbers, which correspond to j, and the two magnetic quantum numbers, m and $\bar{m}$, in an $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ representation.

Explicitly, the eigenfunction of $J^2, J^3, \bar{J}^3$ is given by (see [30] for notation)

$$\Phi^j_{m\bar{m}} = Ce^{-i\nu t}e^{il\phi}\cos^{2j}\mu\sin^l\mu\ {}_2F_1(j+\frac{l+\nu}{2}, j+\frac{l-\nu}{2}, l+1; \sin^2\mu) \tag{8}$$

where $\nu = m+\bar{m}, l = \bar{m}-m$, and the coordinate μ is defined by $\tan\mu = \sinh\rho$. C is a normalization constant, which sets the overlap of $\Phi^j_{m\bar{m}}$ with itself to 1.

The states of particles on AdS_3 are given by (delta function) normalizable states. This forces j into one of two possible regions:

- $j > \frac{1}{2}$, j real—discrete representations $D^\pm_j$

These are representations with highest (or lowest) weight state. They have real j and a spectrum of magnetic quantum numbers m which starts with the highest (lowest) $m = j$, which is annihilated by J^+ (or J^-) and moves down (up) a unit by repeated application of J^- (or J^+). These representations are unitary for $j > \frac{1}{2}$.

- $j = \frac{1+is}{2}$ (s real)—continuous representations C_j^α

  These do not have highest or lowest weight states, and therefore the spectrum of m is unbounded from above and from below. The fractional part of m is preserved by an action of the creation and annihilation operators, and is denoted by α. The spectrum is then $m = \alpha + k$ where k is an arbitrary integer.

A complete basis of normalizable functions on AdS_3 is spanned by the representations $D_j^\pm \times D_j^\pm$ ($j > \frac{1}{2}$) and $C_j^\alpha \times C_j^\alpha$. A state in one of those representations $|j,m\rangle \times |j,\bar{m}\rangle$ can be represented by the function $\Phi_{jm\bar{m}}$ defined above.

2.2 Strings on AdS_3

Quantization of strings on AdS_3 was performed in [9], which we review here. The closed string spectrum found in [9] is the starting point to building boundary states for the D-branes propagating in AdS_3.

For each representation of the zero mode algebra $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ one can construct a module of the Kac-Moody algebra by a repeated application of oscillator modes $J^a_{-n}, n > 0, a = \pm, 3$. We denote such representations by $\hat{D}_j^\pm$ and $\hat{C}_j^\alpha$. These representations were dubbed positive energy representations in [9], as the spectrum of L_0 is bounded from below.

The no-ghost theorem of [3, 9] states that to ensure unitarity (after imposing the Virasoro constraints) one must restrict the spectrum of the discrete representations $\frac{1}{2} < j < \frac{k-1}{2}$. There is no restriction on the spectrum of the continuous representations.

In [9], a new set of representations of the KM algebra was constructed. These are obtained from the above representations by an application of the spectral flow, defined as :

$$J_n^3 \to J_n^3 - \frac{k}{2}\omega\delta_{n,0} \qquad J_n^+ \to J_{n+\omega}^+ \qquad J_n^- \to J_{n-\omega}^- \tag{9}$$

This preserves the KM algebra, and was conjectured in [9] to be a symmetry of the closed string spectrum. The new representations have L_0 unbounded from below, but satisfy a no-ghost theorem, proven in [9].

The complete closed string spectrum then consists of the representations allowed by the no-ghost theorem above, together with all their images under the spectral flow—the so-called flowed representations. We denote such representations by $\hat{D}_j^{\pm,\omega}$ and $\hat{C}_j^{\alpha,\omega}$. Note that one can restrict attention to $\hat{D}_j^{+,\omega}$ only, since $\hat{D}_j^{-,\omega}$ can be generated from it by spectral flow.

2.3 Winding Strings on AdS_3

For later use, we are interested in compactifying time in the Euclidean version of the theory. This will be used as a regulator, as in [9]. The compactification introduces new sectors of closed strings, namely the winding strings. We quantize these strings, following [9].

Any string configuration is represented by a map $g(\sigma,\tau)$ from the worldsheet to spacetime. As shown by [9], the general such map which satisfies the equations of motion can be written as $g = g_+(x^+)g_-(x^-)$, where $x^\pm = \tau \pm \sigma$.

Starting with any such configuration, one can obtain a new one by the transformation:

$$g_+ \to e^{\frac{i}{2}\omega_R x^+ \sigma_2} g_+ \qquad g_- \to e^{\frac{i}{2}\omega_L x^- \sigma_2} g_+ \tag{10}$$

This is equivalent to

$$\begin{aligned} t &\to t + \frac{1}{2}(\omega_L + \omega_R)\tau + \frac{1}{2}(\omega_R - \omega_L)\sigma \\ \phi &\to \phi + \frac{1}{2}(\omega_L + \omega_R)\sigma + \frac{1}{2}(\omega_R - \omega_L)\tau \end{aligned} \tag{11}$$

This action with $\omega_l = \omega_R$ generates spectral flow. If we are working on the universal cover of AdS_3, then the time direction is noncompact. We are then forced to set $\omega_l = \omega_R$ so that fields are single valued.

On the single cover of AdS_3, on the other hand, the time direction is compact with radius 2π. We can then take $\omega_R = -\omega_L = w$. The resulting configuration satisfies:

$$t(\sigma + 2\pi) = t + 2\pi w \tag{12}$$

i.e. the string winds w times around the time direction.

More generally, if we compactify the time direction with the identification $t \equiv t + 4\pi R$, then taking $\omega_R = -\omega_L = 2wR$ generates a string winding w times around the time direction.

The currents on the worldsheet are given by

$$\begin{aligned} J^3 &= k(\partial_+ u + \cosh 2\rho \partial_+ v) \\ J^\pm &= k(\partial_+ \rho \pm i \sinh 2\rho \partial_+ v) e^{\mp 2iu} \\ \bar{J}^3 &= k(\partial_- v + \cosh 2\rho \partial_- u) \\ \bar{J}^\pm &= k(\partial_- \rho \pm i \sinh 2\rho \partial_- u) e^{\mp 2iv} \end{aligned} \tag{13}$$

The effect of creating winding w described above is:

$$\begin{aligned} J^3 &\to J^3 + kwR & \qquad \bar{J}^3 &\to \bar{J}^3 - kwR \\ J^\pm &\to e^{\mp 2iRwx^+} J^\pm & \qquad \bar{J}^\pm &\to e^{\pm 2iRwx^-} \bar{J}^\pm \end{aligned} \tag{14}$$

In terms of the modes, we have:

$$\begin{aligned} J^3_n &\to J^3_n + kwR\delta_{n,0} & \qquad \bar{J}^3_n &\to \bar{J}^3_n - kwR\delta_{n,0} \\ J^\pm_n &\to J^\pm_{n\mp 2wR} & \qquad \bar{J}^\pm_n &\to \bar{J}^\pm_{n\pm 2wR} \end{aligned} \tag{15}$$

These formulae are similar to the ones in [9] for the spectral flow, except that the shifts on the two sides are opposite. Thus the operation here creates winding around the time direction, rather than a long string winding. The stress tensor on the worldsheet is shifted to be:

$$L_0 \to L_0 - 2wRJ_0^3 - kw^2R^2 \qquad\qquad \bar{L}_0 \to \bar{L}_0 + 2wR\bar{J}_0^3 - kw^2R^2 \tag{16}$$

We will also need to consider winding strings for the case of Euclidean AdS_3. This corresponds to a Wick rotation of both t and the worldsheet time τ. All formulae for this case are the same with the replacement $R \to iR$.

2.4 Geometry of AdS_3 D-branes

We turn now to the D-branes in AdS_3. First we review the semiclassical, geometric, analysis performed in [12]. We concentrate on the physical 1-branes in Lorenzian AdS_3[1]. Those were found to wrap conjugacy classes which are generically AdS_2. The conjugacy classes are topologically trivial, but the wrapped branes are prevented from collapsing by a gauge field flux on their worldvolume, as explained in [18]. Since the worldvolume is non-compact, there is no quantization condition on the flux, and consequently any conjugacy class is allowed (at the string tree level).

The 1-branes under consideration wrap twisted (twined) conjugacy classes. These conjugacy classes are defined by having a constant value of $tr(\sigma_1 g)$, where g is an element of SL$(2,\mathbb{R})$ in its parametrization by 2×2 matrices, and σ_1 is the Pauli matrix. The conjugation by the matrix σ_1 generates the unique outer automorphism of SL$(2,\mathbb{R})$. In cylindrical (global) coordinates these are the hypersurfaces:

$$\sinh\psi = \sinh\rho\sin\phi = C \tag{17}$$

where C is some constant. Geometrically, these are static configuration of D-strings connecting antipodal points of AdS_3.

In addition to these 1-branes, one can construct branes wrapping regular conjugacy classes. Those are characterized by $tr(g)$ being a constant, which translates to $\cosh\rho\cos t =$ const. Depending on the value of the constant, the regular conjugacy classes span H_2, the hyperbolic plane, or dS_2 —two dimensional deSitter space. While the former is appropriate to describing instantons in AdS_3, the later were shown to correspond to unphysical D-branes, carrying hyper-critical electric field on their worldvolume. In the following we will restrict our attention to the branes wrapping the twisted conjugacy classes only.

3 Analysis in Conformal Field Theory

3.1 Ishibashi states

We want to extend the semi-classical analysis to a full conformal field theory treatment, by writing the boundary state which describes these D-branes. The condition

[1]That is, worldvolumes which are 1+1 dimensional.

in conformal field theory that the branes wrap the above conjugacy class is [25, 27]

$$(J^a_n + \bar{J}^a_{-n})|B\rangle = 0 \tag{18}$$

where J^a, $\bar{J}^a$ are the Kac-Moody currents on the worldsheet, and $|B\rangle$ is the boundary state.

Cognoscenti may be surprised at the above equation, which is usually associated with regular conjugacy classes, not twisted ones. This small peculiarity follows from our definition of the currents $J^a, \bar{J}^a$. In our notation[2], the condition (18) indeed implies (17). In particular, the branes wrapping the conjugacy classes defined in (17) are static, therefore by energy conservation they couple (linearly) only to zero energy closed string states. In other words, the boundary state for the twined conjugacy class should satisfy $\partial_t|B\rangle = 0$, i.e. $(J^3 + \bar{J}^3)|B\rangle = 0$ which is consistent with (18).

En passant, we note that the branes wrapping regular conjugacy classes are characterized by the condition

$$(J^3 - \bar{J}^3)|B\rangle = (J^+ + \bar{J}^-)|B\rangle = (J^- + \bar{J}^+)|B\rangle = 0 \tag{19}$$

Following [21, 22], one might hope to find a solution to (18) for each module of the chiral algebra. For our purposes it is sufficient to construct Ishibashi states based on continuous representations, $\hat{C}^{\alpha,\omega}_j$, only. Indeed, the branes we discuss couple only to zero energy closed string states. In the discrete representations there are at most finitely many such states, insufficient number to construct a boundary state. It is only the continuous representations that have infinitely many zero energy states, which can then form a coherent state. We conclude then that the boundary state we are seeking has an overlap with the representations $\hat{C}^{\alpha,\omega}_j$ only[3].

In addition, the standard Ishibashi construction of a solution to (18) does not work for the discrete representations. The simplest way to see this is the following: the condition $(J^+_0 + \bar{J_0}^+)|B\rangle = 0$ is usually satisfied by comparing the components of the boundary state $|B\rangle$ with magnetic quantum numbers $(m-1, \bar{m})$ and $(m, \bar{m}-1)$. This generates a recursion relation whose solution is the set of coefficients of the Ishibashi state.

However, here the spectrum of the magnetic quantum numbers is semi-infinite, so some of the states needed for the recursive cancellation are absent from the spectrum. This is then another argument to show that the discrete representations do not make an appearance in our boundary state. We are then only interested in Ishibashi states based on the continuous representations $\hat{C}^{\alpha,\omega}_j$.

Furthermore, we are interested at first in the semiclassical, infinite k limit, so we start by restricting attention to the unflowed representations $\hat{C}^{\alpha}_j$ only. The standard Ishibashi construction does work for the representations $\hat{C}^{\alpha}_j$. Suppose we are given a KM primary $|\Phi_j\rangle$ in the $\hat{C}^{\alpha}_j \times \hat{C}^{\alpha}_j$ representation. By that we mean a state which is

[2] This is not just a matter of notation. In order to use the spectrum of representations in the closed string spectrum as stated in [9], one has to adhere to their notation. Using different notation would entail translating their statements concerning the representation content as well.

[3] This has to be contrasted with the recent discussion in [15]. See also our remarks in the previous footnote.

annihilated by all lowering operators $J^a_n, n > 0$, but not by the zero modes (since C^α_j has no highest weight states). Define[22, 16]

$$|I^j\rangle = \sum_{I,J} M^{-1}_{IJ} J_{-I} \bar{J}_{-J} |\Phi^j\rangle \tag{20}$$

Here I, J are ordered strings of indices $(n_1, a_1) \cdots (n_r, a_r)$, and

$$J_I = J^{a_1}_{n_1} \cdots J^{a_r}_{n_r}. \tag{21}$$

For later convenience we choose an ordering such that all zero modes act from the left. This sums over all the descendants in the KM module, with the normalization defined as

$$M_{IJ} = \langle \Phi^j | J_I \, J_{-J} | \Phi^j \rangle \tag{22}$$

M_{IJ} is invertible for any KM module. (For degenerate modules, one has to mod out by the null vectors). It is easy to see that $|I^j\rangle$ satisfies (18) by showing that $(J_n + \bar{J}_{-n})|I^j\rangle$ is orthogonal to all states in the module based on $|\Phi^j\rangle$.

We will also need to construct Ishibashi states when the moding of the oscillators is shifted, as in the winding string sector. The formulae are very similar to the unshifted case, and are discussed in e.g. [19].

3.2 Overlaps of Ishibashi states-a difficulty

The boundary states $|I^j\rangle$ based on primaries in the $\hat{C}^\alpha_j \times \hat{C}^\alpha_j$ representations are the building blocks of the desired boundary state. The physical boundary state is required to satisfy Cardy's conditions [20], which guarantee the existence of open string quantization of the system.

Here we point out a difficulty in constructing a physical boundary state, which results from the fact that the representation of the zero mode algebra, C^α_j, is infinite dimensional. The overlaps between Ishibashi states are as usual the characters of the corresponding representations:

$$\langle I^j | q^{L_0 + \bar{L}_0 - \frac{c}{12}} | I^j \rangle = Tr_j (q^{2L_0 - \frac{c}{12}}) = \chi_j(q^2) \tag{23}$$

where χ_j is the character in the representation based on Φ^j. This character diverges since the character involves a sum over all magnetic quantum numbers m (through descendants obtained by application of the zero mode operators $J^\pm_0$), and L_0 is independent of m. This gives a divergence which needs regulating. We will find below a way to represent this divergence which makes regularization straightforward.

A regularization which works for the discrete representation is replacing the character above by $tr_j(q^{2L_0 - \frac{c}{12}} \, e^{2\pi i \theta J^3_0})$, where θ is used as a regulator. This works well for the discrete representation, where it replaces the divergence by a well behaved modular functioni [5]. For the continuous representations, however, this regularization yields a result proportional to $\delta(\theta)$, which is an awkward object to manipulate.

We emphasize that the difficulty has to do with the zero modes, and not with the string oscillator modes. It will be already present when quantizing a particle on

AdS_3. In an attempt to separate the stringy aspects from the infrared aspects of the boundary states, we define the following coherent state:

$$|I^j_{m\bar{m}}\rangle = \widetilde{\sum_{I,J}} M^{-1}_{IJ} J_{-I} \bar{J}_{-J} |\Phi^j_{m\bar{m}}\rangle \tag{24}$$

Here $|\Phi^j_{m\bar{m}}\rangle$ is annihilated by all oscillator modes except for the zero modes (as above), and has magnetic quantum numbers m and $\bar{m}$. The sum $\widetilde{\sum}$ over descendants here is defined to exclude any action by the zero modes $J_0^{\pm}$. It is clear by definition that:

$$|I^j\rangle = \sum_m |I^j_{m,-m}\rangle \tag{25}$$

The overlap of these coherent states is finite. It is possible to exhibit this overlap as a product of a contribution from the primaries and a contribution of the stringy oscillators:

$$\langle I^j_{m\bar{m}}| q^{L_0+\bar{L}_0-\frac{c}{12}} e^{\pi i\theta(J_0^3-\bar{J}_0^3)} |I^j_{m\bar{m}}\rangle = \frac{\langle \Phi^j_{m\bar{m}}| q^{L_0+\bar{L}_0-\frac{c}{12}} e^{\pi i\theta(J_0^3-\bar{J}_0^3)} |\Phi^j_{m\bar{m}}\rangle}{\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n}e^{2\pi i\theta})(1-q^{2n}e^{-2\pi i\theta})} \tag{26}$$

The overlap of the primaries written above is

$$\langle \Phi^j_{m\bar{m}}| q^{L_0+\bar{L}_0-\frac{c}{12}} e^{\pi i\theta(J_0^3-\bar{J}_0^3)} |\Phi^j_{m\bar{m}}\rangle = q^{\left(-\frac{2j(j-1)}{k-2}-\frac{k}{4(k-2)}\right)} e^{\pi i\theta(m-\bar{m})} \tag{27}$$

3.3 From geometry to CFT

The full CFT description requires us to find a linear combination of Ishibashi states, which satisfies Cardy's condition. This is rendered somewhat difficult because of the divergences discussed in subsection (3.2). We will see now how the geometry of the AdS_3 branes (discussed in subsection (2.4)) helps us to understand the boundary state description of the D-branes.

The connection between the boundary state and the D-brane geometry is well-known in the study of the $SU(2)$ WZW model. This is most clearly explained in Appendix B of [26]; we review their analysis here.

Define a graviton wavepacket $|x\rangle$ localised at a point x on the group manifold, i.e.

$$\langle x|\Phi\rangle = \Phi(x) \tag{28}$$

In the semiclassical limit, the D-brane is localized at the conjugacy class $\psi = \psi_0$. Hence the overlap of the D-brane with the graviton wavepacket described above should also be localized on the conjugacy class i.e.

$$\lim_{k\to\infty} \langle x|B\rangle = \frac{f(\psi_0)}{\cosh\psi_0}\delta(\psi-\psi_0) \tag{29}$$

where $f(\psi_0)$ is a function proportional to the tension of the brane.

In [27, 26], it was shown that a formula of the above form indeed holds for the branes in the $SU(2)$ WZW model i.e. in the large k limit, the boundary states are found to be located on conjugacy classes.

We will reverse the procedure for the SL(2, IR) case; given the geometry of the brane, we will obtain information about the boundary state.

The boundary state is in general a linear superposition of Ishibashi states

$$|B\rangle = \sum_{jm\bar{m}} c^j_{m\bar{m}} |I^j_{m\bar{m}}\rangle \tag{30}$$

where $|I^j_{m\bar{m}}\rangle$ is the coherent state based on the primary $\Phi^j_{m\bar{m}}$ as in (24). The $c^j_{m\bar{m}}$ are constants which are the main data required for specifying the boundary state. We shall determine the coefficients $c^j_{m\bar{m}}$ (to leading order in $\frac{1}{k}$) by using equation (29)[4].

Combining the various equations, we see that the required equation for the coefficients is

$$\sum_{jm\bar{m}} c^j_{m\bar{m}}(\psi_0)\Phi^j_{m\bar{m}}(\psi,\chi,t) = \frac{f(\psi_0)}{\cosh\psi_0}\delta(\psi-\psi_0) \tag{31}$$

where we have exhibited the dependence of the coefficients $c^j_{m\bar{m}}$ on the conjugacy class, parametrized by its location ψ_0.

Furthermore, the construction of the Ishibashi state tells us that all $\Phi^j_{m\bar{m}}$ with $m+\bar{m}=0$ contribute equally, i.e.

$$c^j_{m\bar{m}} = c^j\delta_{m+\bar{m},0} \tag{32}$$

So the boundary state can be written

$$|B\rangle = \sum_j c^j \sum_m |I^j_{m,-m}\rangle \tag{33}$$

and equation (31) simplifies to

$$\sum_j c^j(\psi_0)\sum_m \Phi^j_{m,-m}(\psi,\chi,t) = \frac{f(\psi_0)}{\cosh\psi_0}\delta(\psi-\psi_0) \tag{34}$$

Our task is now to invert this equation and obtain the coefficients $c^j(\psi_0)$. Note that the overlap of two such boundary states is still divergent, since it includes the following factor, resulting from the overlap of primaries:

$$\langle B|q^{L_0}|B\rangle \propto \sum_j |c^j|^2 \sum_m \langle\Phi^j_{m,-m}|q^{L_0}|\Phi^j_{m,-m}\rangle = \sum_j |c^j|^2 \sum_m q^{-j(j-1)/(k-2)} \tag{35}$$

Since we have an infinite sum over m, this expression diverges.

This divergence is not regulated by separating the branes e.g. if we separate the branes by rotating one of them through an angle θ, the overlap contains the factor

$$\langle B|q^{L_0}e^{i\theta(J^3_0-\bar{J}^3_0)}|B\rangle \propto \sum_j |c^j|^2 \sum_m q^{-j(j-1)/(k-2)}e^{2im\theta} = \sum_j |c^j|^2 q^{-j(j-1)/(k-2)}\delta(\theta) \tag{36}$$

What we have found is exactly the divergence discussed earlier, where we found that the characters of the continuous representations were ill defined. However, we presented here the divergence in a form amenable to regularization, which we now perform.

[4] In the SU(2) case, the coefficients were already known from the work of [20, 21].

3.4 The correct approach

As we have seen, our equation (31) has reduced to

$$\sum_j c^j(\psi_0) \sum_m \Phi^j_{m,-m}(\psi, \chi, t) = \frac{f(\psi_0)}{\cosh \psi_0} \delta(\psi - \psi_0) \tag{37}$$

So instead of considering each $\Phi^j_{m,-m}$, we see that the only relevant combination we need to consider is

$$\Phi^j = \sum_m \Phi^j_{m,-m} \tag{38}$$

Surprisingly, we see that if we use Φ^j, all our problems can be resolved!

First, to determine Φ^j as a function of (ψ, χ, t), note that from the definition above, we have $(J^a + \bar{J}^a)\Phi^j = 0$.

Using (5),

$$(J^3 + \bar{J}^3)\Phi^j = 0 \quad \Rightarrow \quad \partial_t \Phi^j = 0. \tag{39}$$

The equation $(J^+ + \bar{J}^+)\Phi^j = 0$ can then be written

$$\begin{aligned} e^{-2iu} \left(\frac{\cosh 2\rho}{\sinh 2\rho} \partial_u - \frac{1}{\sinh 2\rho} \partial_v + \frac{i}{2} \partial_\rho \right) \Phi^j(\psi, \chi) \\ + e^{-2iv} \left(\frac{\cosh 2\rho}{\sinh 2\rho} \partial_v - \frac{1}{\sinh 2\rho} \partial_u + \frac{i}{2} \partial_\rho \right) \Phi^j(\psi, \chi) = 0 \end{aligned} \tag{40}$$

which reduces to

$$\partial_\chi \Phi^j(\psi, \chi) = 0 \tag{41}$$

Hence we find that Φ^j is a function of ψ alone.

Since Φ^j was defined as a linear combination of $\Phi^j_{m,-m}$, which are all eigenfunctions of the Casimir with eigenvalue $4j(j-1)$, the same holds true for Φ^j i.e.

$$\Box \Phi^j = 4j(j-1)\Phi^j \tag{42}$$

Since Φ^j is a function of ψ alone, this equation becomes

$$\partial_\psi^2 \Phi^j + \frac{2\sinh\psi}{\cosh\psi} \partial_\psi \Phi^j = 4j(j-1)\Phi^j \tag{43}$$

The two independent solutions to this differential equation are [5]

$$\Phi^j = \frac{e^{\pm(2j-1)\psi}}{\cosh\psi} \tag{44}$$

[5] The Φ^j here are related to the definition in (38) by an overall constant.

Note that if $(2j-1) > 0$ then Φ^j diverges either at $\psi = \infty$ or $\psi = -\infty$. If we want Φ^j to be normalizable, we must take $(2j-1) = is$ with s real. Comparing with the closed string spectrum in section 2, this implies that Φ^j is a combination of states in the continuous representations alone, and discrete states do not have any overlap with the D-brane. This is consistent with the analysis in section 3.

Now that we know Φ^j, we return to equation (31),

$$\sum_j c^j(\psi_0)\Phi^j(\psi) = \frac{f(\psi_0)}{\cosh\psi_0}\delta(\psi - \psi_0) \tag{45}$$

which we can write as

$$\sum_s c^s(\psi_0)\frac{e^{is\psi}}{\cosh\psi} = \frac{f(\psi_0)}{\cosh\psi_0}\delta(\psi - \psi_0) \tag{46}$$

The obvious solution is that $c^s(\psi_0)$ is proportional to $e^{-is\psi_0}$. The sum over s will then yield a delta function in $(\psi - \psi_0)$. The function $f(\psi_0)$ cannot be determined at this level. We will later use Cardy's condition to determine it.

Thus we have found the boundary state in the semiclassical approximation to be

$$|B\rangle = f(\psi_0)\sum_s e^{-is\psi_0}|I^s\rangle \tag{47}$$

where I_s is the Ishibashi state (as defined in (24)) based on the primary $|\Phi^s\rangle$, satisfying $\langle x|\Phi^s\rangle = \frac{e^{is\psi}}{\cosh\psi}$.

We now turn into a calculation of the overlap of different boundary states. This exhibits the regularization of the above discussed divergence, and will serve as the basis of comparison to the open string sector, and to the discussion of Cardy's condition that follows.

4 Cardy condition 1—Overlaps of branes

We will compute four different overlaps in this section.

a) The overlap of two branes, one located at $\psi = \psi_0$, and the other at $\psi = \tilde{\psi}_0$.

b) The overlap after one of the branes in (a) is rotated through an angle ϕ_0.

c) The overlap of the branes in (a), when they exchange winding modes.

d) The overlap of the branes in (b), when they exchange winding modes.

4.1 Notation

We will be considering two D-branes labeled by ψ_0 and $\tilde{\psi}_0$, so that

$$|B\rangle = f(\psi_0)\sum_s e^{-is\psi_0}|I^s\rangle \qquad |\tilde{B}\rangle = f(\tilde{\psi}_0)\sum_s e^{-is\tilde{\psi}_0}|I^s\rangle \tag{48}$$

Overlaps of boundary states are defined through $\langle\tilde{B}|q^{L_0+\bar{L}_0-\frac{c}{12}}|B\rangle$, which represents the annulus diagram as calculated in the closed string sector. When computing the overlap, it will prove convenient at intermediate stages to restrict to the low energy limit, ignoring stringy oscillators. For this purpose we define:

$$|C\rangle = \sum_s e^{-is\psi_0}|\Phi^s\rangle \qquad |\tilde{C}\rangle = \sum_s e^{-is\tilde{\psi}_0}|\Phi^s\rangle \tag{49}$$

This only has overlap with primaries. We can then relate overlaps of $|B\rangle$ and $|C\rangle$ by using (26).

We will also need the formula:

$$\left(L_0 + \bar{L}_0 - \frac{c}{12}\right)|\Phi^s\rangle = \left(\frac{-2j(j-1)}{k-2} - \frac{k}{4(k-2)}\right)|\Phi^s\rangle = \left(\frac{s^2}{2(k-2)} - \frac{1}{4}\right)|\Phi^s\rangle \tag{50}$$

Also we define $q = e^{i\pi\tau}$.

4.2 Overlaps of parallel branes

The overlap of the branes described above is

$$\langle\tilde{C}|q^{L_0+\bar{L}_0-\frac{c}{12}}|C\rangle = q^{-\frac{1}{4}}\sqrt{\frac{k-2}{2i\pi^2\tau}}\exp\left(\frac{(\tilde{\psi}_0-\psi_0)^2(k-2)}{2i\pi\tau}\right) \tag{51}$$

Hence using equation (26)

$$\langle\tilde{B}|q^{L_0+\bar{L}_0-\frac{c}{12}}|B\rangle = f(\psi_0)f(\tilde{\psi}_0)\sqrt{\frac{k-2}{2i\pi^2\tau}}\exp\left(\frac{(\tilde{\psi}_0-\psi_0)^2(k-2)}{2i\pi\tau}\right)\frac{2\pi}{\theta_1'(0,\tau)} \tag{52}$$

4.3 Overlaps of rotated branes

Consider now the case when the two branes are rotated with respect to each other in the ϕ direction. If a brane is located at $\sinh\psi = \sinh\rho\sin\phi = \sinh\psi_0$, then after rotation it will be located at $\sinh\rho\sin(\phi-\phi_0) = \sinh\psi_0$ i.e.

$$\sinh\psi\cos\phi_0 + \cosh\psi\sinh\chi\sin\phi_0 = \sinh\psi_0 \tag{53}$$

(cf. coordinate system at equation (2).)

We will compute the overlap between the above described brane and a brane located at $\sinh\psi = \sinh\tilde{\psi}_0$.

From a CFT point of view we are computing:

$$\langle\tilde{B}|q^{L_0+\bar{L}_0-\frac{c}{12}}e^{-i\phi_0(J_0^3-\bar{J}_0^3)}|B\rangle \tag{54}$$

Here J_0^3 and $\bar{J}_0^3$ are the zero modes on the worldsheet of the currents J^3 and $\bar{J}^3$.

We compute first

$$\langle\tilde{C}|q^{L_0+\bar{L}_0-\frac{c}{12}}e^{-i\phi_0(J_0^3-\bar{J}_0^3)}|C\rangle = \int dx\ \langle\tilde{C}|q^{L_0+\bar{L}_0-\frac{c}{12}}|x\rangle\langle x|e^{-i\phi_0(J_0^3-\bar{J}_0^3)}|C\rangle \tag{55}$$

Here $|x\rangle$ is a complete basis of position eigenstates and $\int dx|x\rangle\langle x| = 1$.

By definition of the rotated brane:

$$\langle x|e^{-i\phi_0(J_0^3-\bar{J}_0^3)}|C\rangle = \delta(\sinh\psi\cos\phi_0 + \cosh\psi\sinh\chi\sin\phi_0 - \sinh\psi_0) \tag{56}$$

From the boundary state description of $\tilde{C}$

$$\langle\tilde{C}|q^{L_0+\bar{L}_0-\frac{c}{12}}|x\rangle = \int \frac{ds}{2\pi}q^{-\frac{1}{4}}q^{\left(\frac{s^2}{2(k-2)}\right)}e^{i\tilde{\psi}_0 s}\frac{e^{-i\tilde{\psi}s}}{\cosh\psi} \tag{57}$$

The overlap $\langle\tilde{C}|q^{L_0+\bar{L}_0-\frac{c}{12}}e^{-i\phi_0(J_0^3-\bar{J}_0^3)}|C\rangle$ is then

$$\begin{aligned}
&k^{3/2}\int dtd\psi d\chi\cosh^2\psi\cosh\chi \\
&\qquad\times\ \delta(\sinh\psi\cos\phi_0 + \cosh\psi\sinh\chi\sin\phi_0 - \sinh\psi_0) \\
&\qquad\qquad\times\int \frac{ds}{2\pi}q^{-\frac{1}{4}}q^{\left(\frac{s^2}{2(k-2)}\right)}e^{i\tilde{\psi}_0 s}\frac{e^{-i\tilde{\psi}s}}{\cosh\psi} \\
&= \frac{k^{3/2}R_t}{|\sin\phi_0|}\int dsq^{-\frac{1}{4}}q^{\left(\frac{s^2}{2(k-2)}\right)}\delta(s) \\
&= \frac{k^{3/2}R_t}{|\sin\phi_0|}q^{-\frac{1}{4}}
\end{aligned} \tag{58}$$

where we have assumed the time direction to be compact with radius R_t.

So, using (26), the full overlap is

$$\langle\tilde{B}|q^{L_0+\bar{L}_0-\frac{c}{12}}e^{-i\phi_0(J_0^3-\bar{J}_0^3)}|B\rangle = \frac{2f(\psi_0)f(\tilde{\psi}_0)k^{3/2}R_t}{|\theta_1\left(\frac{\phi_0}{\pi},\tau\right)|} \tag{59}$$

Note that only the $s = 0$ term contributes.

4.4 Overlaps of parallel branes-winding sectors

Cardy's condition requires us to interpret an overlap of branes as a modular transformation of an open string partition function. In usual cases, (e.g. $SU(2)$ WZW models), this requires the modular transformation to be a linear combination of open string characters with positive integer coefficients.

Unfortunately, the $\mathrm{SL}(2,\mathbb{R})$ model has a continuum of states (the long string states [4]) and it is difficult to extract the discrete states out of this continuum. A method for doing this was proposed in [9]. For our purposes, duplication of this method requires us to compactify the time direction, and compute overlaps of branes as a function of the radius of the time direction.

The nontrivial dependence on the radius occurs because now there are new states in the theory with which the boundary state has overlap, viz. strings winding around the time direction. Exchange of such winding modes leads to new terms in the overlap of the boundary states.

In general, the boundary state can now be written

$$|B\rangle = \sum_w |B_w\rangle \tag{60}$$

where $|B_w\rangle$ is the part of the boundary state $|B\rangle$ having overlap with strings of winding number w.

The overlap is now a sum of terms

$$\langle B|q^{L_0+\bar{L}_0-\frac{c}{12}}|B\rangle = \sum_w \langle B_w|q^{L_0+\bar{L}_0-\frac{c}{12}}|B_w\rangle \tag{61}$$

To compute this overlap, we note that the single cover of $\mathrm{SL}(2,\mathbb{R})$ has a compact timelike direction. There is a group transformation that transforms a string winding w times around this compact timelike directon to an unwound string (as discussed in (2.3)). This is very similar to the transformation that produces spectral flow. Since spectral flow is believed to be an exact symmetry, it is plausible that is other transformation is also an exact symmetry, and we conjecture that it is. This transformation should leave the boundary state unchanged, and hence we should have $|B_w\rangle \to |B_0\rangle$.

From equation (16), the transformation also acts on the energy as a shift

$$(L_0 + \bar{L}_0) \to (L_0 + \bar{L}_0)' = (L_0 + \bar{L}_0) - (2wR)(J_0^3 - \bar{J}_0^3) - 2kw^2R^2 \tag{62}$$

This together implies

$$\langle \tilde{B}_w|q^{L_0+\bar{L}_0-\frac{c}{12}}|B_w\rangle = q^{-2kw^2R^2}\langle \tilde{B}_0|q^{L_0+\bar{L}_0-\frac{c}{12}}e^{-i\phi_0(J_0^3-\bar{J}_0^3)}|B_0\rangle \tag{63}$$

where $\phi_0 = 2\pi wR\tau$.

This overlap was already calculated in the previous subsection. We find

$$\langle \tilde{B}_w|q^{L_0+\bar{L}_0-\frac{c}{12}}|B_w\rangle = q^{-2kw^2R^2}(8\pi k^{3/2}R)\frac{f(\psi_0)f(\tilde{\psi}_0)}{|\theta_1\left(2wR\tau,\tau\right)|} \tag{64}$$

Since we get the same contribution from w and $-w$, we can restrict the sum to run over the positive integers only. Then the overlap is

$$\sum_{w=1}^{\infty} q^{-2kw^2R^2}(16\pi k^{3/2}R)\frac{f(\psi_0)f(\tilde{\psi}_0)}{|\theta_1\left(2wR\tau,\tau\right)|} \tag{65}$$

This is the overlap of branes in Lorentzian AdS_3. In Euclidean AdS_3, the overlap is obtained by replacing R by iR.

4.5 Overlaps of rotated branes (winding sector)

Finally, one can compute the overlaps of relatively rotated branes in the winding sector. This is a simple combination of the two previous subsections. The result is:

$$\langle \tilde{B}|q^{L_0+\bar{L}_0-\frac{c}{12}}e^{-i\phi_0(J_0^3-\bar{J}_0^3)}|B\rangle = \sum_{w=1}^{\infty} q^{-2kw^2R^2}f(\psi_0)f(\tilde{\psi}_0)\frac{16\pi k^{3/2}R}{|\theta_1(2wR\tau+\frac{\phi_0}{\pi},\tau)|} \tag{66}$$

5 Some digressions

5.1 The vanishing divergences

We have obtained a finite answer for the overlap of the boundary states. Where did the divergences in the characters disappear?

It turns out that these divergences are hidden in the infinite volume of the D-branes. We found the D-brane is a source for Φ^j, which is a constant over the conjugacy class. The overlap $\langle \Phi^j | \Phi^j \rangle$ is therefore proportional to the volume squared of the conjugacy class, and hence divergent.

Once the divergence is presented this way, it is easy to regulate. One normalizes the boundary state by a factor $\frac{1}{\sqrt{V}}$, where V is the volume. The resulting overlap then scales like the volume, which is the correct scaling for the open string partition function. On the other hand the overlap of the rotated branes is now finite, as it should be since the open strings are now fixed at the intersection point. Thus the delta function found earlier for the rotated branes is now regulated as well.

Note that the field Φ^j we found is strictly speaking not a normalizable mode, since its overlap diverges. This is because this field is an infinite sum of normalizable modes. (Hence the boundary state only couples to normalizable modes.)

5.2 Relation to the $SU(2)$ Boundary States

The $SL(2,R)$ group manifold is related to the $SU(2)$ group manifold by analytic continuation. We might expect the boundary states to be related this way. This is not the case. For example, there is an analogue of the field Φ^j in the $SU(2)$ case, which is equal to $\frac{\sin((2j+1)\tilde{\psi})}{\sin\tilde{\psi}}$. On the other hand, in the $\mathrm{SL}(2,\mathbb{R})$ case, we had two independent solutions for each j, which were $\Phi^j = \frac{e^{\pm(2j-1)\psi}}{\sinh\psi}$.

The reason for the difference is that in the $SL(2,R)$ case there are two continuous representations, with $j = \frac{1}{2} \pm is$, having the same Casimir, whereas in the $SU(2)$ case there is only one representation for each Casimir.

This appears to be a generic difference between compact and non-compact WZW models. One important effect of this is that we can completely localize the AdS_3 branes, whereas the $SU(2)$ branes are not fully localized on the conjugacy classes for finite k.

Furthermore, if the $\mathrm{SL}(2,\mathbb{R})$ theory had contained finite dimensional representations, then boudary states built on these representations would have been very similar to $SU(2)$ boundary states [23].

5.3 Spectral Flow

So far we have considered only the unflowed representations. We turn now to the flowed representations—those constructed by acting with spectral flow on the positive energy representations. We argue here that flowed representations do not contribute to the boundary state.

Intuitively, the flowed representations correspond to states of long strings winding near the boundary of AdS_3. The D-branes do not wind near the boundary, so classical long string configurations cannot be absorbed by the branes.

Nevertheless, the long string winding number is not conserved when the string moves into the bulk, so there is some possibility of absorption by the brane. This seems implausible to us, since locally the brane we constructed looks like a flat brane in flat space. The overlap with the unflowed representations accounts for all the states we need to reconstruct this flat space limit. An overlap with the flowed representation is then possible only if there is some subtlety in the flat space (large k) limit.

The above heuristic arguments are only semi-classical. The most convincing argument is our ability to construct an exact Cardy state without any recourse to flowed representations. This argument relies on an exact CFT analysis and therefore is not limited to the semiclassical regime.

Two more comments are in order. First, the restriction to unflowed representations does not truncate the spectrum (as it would for discrete represenatations), and therefore there is no problem with reproducing the flat space boundary state. Secondly, the open string spectrum can and does carry long string winding number. Indeed, the modular transform of the Cardy state we construct reproduces the open string result of [13].

5.4 Internal CFT and ghosts

To construct a complete bosonic critical string theory containing an AdS_3 factor we need to add an extra matter CFT as well as b, c ghost system. An example of an extra matter CFT is $S^3 \times T^{20}$.

The boundary states and their overlaps factorize into contributions from the 3 separate CFT factors. We need certain properties about the modular transformation of the overlaps of the matter CFT and the ghost sector.

Under $\tau \to -\frac{1}{\tilde{\tau}}$ we have, for the ghost sector (up to overall constants)

$$ {}_{gh}\langle B|q^{L_0+\bar{L}_0-\frac{c}{12}}|B\rangle_{gh} \quad \to \quad \frac{\tilde{q}^{1/6}}{\tilde{\tau}}\prod_n [1-\tilde{q}^{2n}]^2 \tag{67} $$

Secondly, we will assume that we have already found a Cardy state in the internal CFT, i.e. the modular transform of that part of the boundary state yields a well defined open string partition function.

The total central charge of the internal CFT on the open string is $c_{int} = 26 - c_{AdS_3} = 26 - \frac{3k}{k-2}$. The modular transform of the part of the boundary state depending on the internal CFT M is then of the form (up to overall constants)

$$ {}_M\langle B|q^{L_0+\bar{L}_0-\frac{c}{12}}|B\rangle_M \quad \to \quad \tilde{q}^{-\frac{c_{int}}{12}}\sum_h D(h)\tilde{q}^{2h} \tag{68} $$

where h is the weight in the open string conformal field theory, and $D(h)$ is the degeneracy at weight h.

6 Cardy condition 2—Open string partition function

6.1 $\psi_0 = 0$

We now modular transform the brane overlaps to obtain the open string partition function. We start by looking at the overlap in the winding sector of two branes, both located at $\psi_0 = 0$. We can compare this to the computation in Appendix B of [13].

The open string partition function, calculated in Euclidean AdS_3 is [13]

$$Z(\beta) = \frac{\beta\sqrt{k-2}}{4\pi\sqrt{2}} \sum_{m=1}^{\infty} \frac{1}{t^{3/2}} e^{2\pi t(1-\frac{1}{4(k-2)})} \sum_h D(h) e^{-2\pi t h} \\ \times \frac{e^{-(k-2)\frac{\beta^2 m^2}{8\pi t}}}{\sinh(\frac{\beta m}{2})} \prod_{n=1}^{\infty} \frac{|1-e^{-2\pi t n}|}{|(1-e^{-2\pi t n+\beta m})(1-e^{-2\pi t n-\beta m})|} \tag{69}$$

On the other hand, we found that the overlap of branes in Euclidean AdS_3 to be

$$16\pi i k^{3/2} R f(\psi_0) f(\tilde{\psi_0}) \sum_{w=1}^{\infty} \frac{q^{2kw^2R^2}}{|\theta_1(2iwR\tau,\tau)|} \tag{70}$$

Defining $\beta = 4\pi R, \tilde{\tau} = it, \tilde{q} = e^{i\pi\tilde{\tau}}$, the modular transform ($\tau = -\frac{1}{\tilde{\tau}}$) of this expression is

$$Z = 4f^2(0)k^{3/2} \sum_{w=1}^{\infty} \frac{e^{-(k-2)\frac{\beta^2 w^2}{8\pi t}} e^{\frac{\pi t}{4}}(\beta)}{\sqrt{t}\sinh(\frac{\beta w}{2})|\prod_{n=1}^{\infty}(1-\tilde{q}^{2n})(1-\tilde{q}^{2n}e^{-\beta w})(1-\tilde{q}^{2n}e^{\beta w})|} \tag{71}$$

The full overlap is then found by multiplying the above expression by the factors coming from the ghost sector and the internal matter sector (67, 68). We see that we get exact agreement with the result from the open string sector (once $f(0)$ is chosen to match the overall constants).

6.2 $\psi_0 \neq 0$

Now we look at the overlap between two branes labeled by ψ_0. This overlap is identical to the one of branes located at $\psi_0 = 0$, except for an additional factor $\frac{f^2(\psi_0)}{f^2(0)}$. On the open string side the partition function picks up the same factor. This means that the degeneracies of open string states is multiplied by that factor. Since ψ_0 is a continuous variable, and the degeneracies are integral, we conclude that $\frac{f^2(\psi_0)}{f^2(0)} = 1$ (to all orders in $\frac{1}{k}$).

This also means that the open string spectrum on all branes is identical. We discuss this further below.

We can hence finally write the complete boundary state as

$$|B\rangle_{tot} = T\ |B\rangle|B\rangle_M|B\rangle_{gh} \tag{72}$$

where T is an overall k dependent factor, which represents the tension of the boundary state (for flat space, the value of T was found in [28]). $|B\rangle$ was defined in (47), and $|B\rangle_M, |B\rangle_{gh}$ are the boundary states for the internal CFT and the ghost sector.

6.3 Rotated branes

The overlaps of rotated branes yield an interesting structure. The modular transform in the zero winding sector is (up to an overall constant)

$$\frac{\tilde{q}^{-\frac{\phi_0^2}{\pi^2}}}{\sqrt{\tilde{\tau}}\ \theta_1(-\frac{\phi_0\tilde{\tau}}{\pi}, \tilde{\tau})} \tag{73}$$

This is very similar to the overlap in the closed string sector with winding. This suggests that the open strings are now described by a shifted stress tensor:

$$L_0 \to L_0 + \frac{\phi_0}{\pi}(J^3 - \bar{J}^3) - \frac{\phi_0^2}{\pi^2} \tag{74}$$

It would be very interesting to see this shift directly in the open string side. The quantization of the open string is subtle, since one has to define the WZ term on worldsheet with boundaries.

7 Discussion and conclusion

Our calculations so far are valid only in the large k limit, where we have a geometrical description. Can we go beyond this limit? We argue now that it is plausible that the boundary state described here is exact for all k.

At first our result, that the open string spectrum is the same on all the branes, might seem surprising. One might expect smaller branes to have a different spectrum than the larger branes.

However, this can be anticipated based on [12]. It turns out that the open string metric, in the sense of [31], is the same on all branes once the electric field is taken into account. This provides some evidence that the open string spectrum is indeed universal.

Secondly, in [13] it was argued that the density of states of long strings with odd spectral flow was different for different branes. This was argued based on semiclassical quantization of the open string, which is valid to leading order in k. At large k, our results are in agreement with [13]. Indeed, the above mentioned difference in densities is subleading in $\frac{1}{k}$.

Moving to finite k, we argue that the boundary state is exact (upto an overall k dependent factor). As we have seen, the boundary states satisfy Cardy's condition for any k. Indeed, using our boundary states localized at any value of ψ_0, we reproduce the result of [13] for the open string sector. As argued in [13], this expression has a sensible interpretation in the open string sector for all values of k.

In a compact CFT, this would be tantamount to a proof that our boundary state is exact. However, in the noncompact case Cardy's condition is not as strong, due to the fact that there is a continuum of states in the open string spectrum. It may be possible then to modify our boundary state at finite k such that only the density of long string states in affected. This cannot be ruled out based on Cardy's condition alone, though we find the scenario implausible.

In [13] a two brane system, located at ψ_0 and $-\psi_0$ was also discussed. This system appears not to have such subtleties, and the spectrum found in [13] was indeed independent of ψ_0. This is in complete agreement with our analysis.

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9 Appendix

AdS_3 is the universal cover of the group manifold $\mathrm{SL}(2,\mathbb{R})$. A convenient parametrization of a $\mathrm{SL}(2,\mathbb{R})$ element is provided by the Euler angles:

$$\begin{aligned} g &= e^{iu\sigma_2} e^{\rho\sigma_3} e^{iv\sigma_2} = \\ &= \begin{pmatrix} \cos t\cosh\rho + \cos\phi\sinh\rho & \sin t\cosh\rho - \sin\phi\sinh\rho \\ -\sin t\cosh\rho - \sin\phi\sinh\rho & \cos t\cosh\rho - \cos\phi\sinh\rho \end{pmatrix} \end{aligned} \tag{75}$$

where $\sigma_i, i = 1, 2, 3$ are the Pauli matrices, and we define:

$$u = \frac{1}{2}(t+\phi) \qquad v = \frac{1}{2}(t-\phi) \tag{76}$$

These are the cylindrical (global) coordinates of AdS_3. The metric is then:

$$ds^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\phi^2 \tag{77}$$

We will often find it useful to replace the global coordinate system (ρ, ϕ, t) by the coordinates (ψ, χ, t), with:

$$\sinh\psi = \sinh\rho\sin\phi \qquad \cosh\psi\sinh\chi = -\sinh\rho\cos\phi \tag{78}$$

The AdS_3 metric in these coordinates is

$$ds^2 = d\psi^2 + \cosh^2\psi(-\cosh^2\chi dt^2 + d\chi^2) \tag{79}$$

Another parametrization of a group element is:

$$g = \begin{pmatrix} X^0 + X^1 & X^2 + X^3 \\ X^2 - X^3 & X^0 - X^1 \end{pmatrix} \tag{80}$$

with

$$(X^0)^2 + (X^3)^2 - (X^2)^2 - (X^1)^2 = 1 \tag{81}$$

This exhibits AdS_3 as a 3-dimensional hyperboloid embedded in $R^{(2,2)}$. The relation between this parametrization and the global coordinates above is:

$$X^0 + iX^3 = \cosh\rho e^{it} \qquad\qquad X^1 + iX^2 = \sinh\rho e^{i\phi} \tag{82}$$

Euclidean AdS_3 is obtained by Wick rotating the global time $t = i\tau$. The metric on the Euclidean AdS_3 is therefore:

$$ds^2 = \cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\phi^2 \tag{83}$$

It is sometime convenient to choose a different parametrization of the Euclidean AdS_3. In the so-called Poincare coordinates the metric is:

$$ds^2 = \frac{1}{r^2}\left(dr^2 + dx^2 + d\tilde{\tau}^2\right) \tag{84}$$

In the hyperboloid representation of AdS_3, the isometry group of the embedding space $R^{(2,2)}$ is generated by the currents:

$$J^{ij} = X^i \frac{\partial}{\partial X^j} - X^j \frac{\partial}{\partial X^i} \tag{85}$$

The $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ generators are then:

$$\begin{aligned} J_1 &= \frac{1}{2}(J_{01} + J_{23}) \qquad & J_2 &= \frac{1}{2}(J_{02} - J_{13}) \qquad & J_0 &= \frac{1}{2}(J_{12} + J_{03}) \\ \bar{J}_1 &= \frac{1}{2}(J_{01} - J_{23}) \qquad & \bar{J}_2 &= \frac{1}{2}(J_{02} + J_{13}) \qquad & \bar{J}_0 &= \frac{1}{2}(J_{12} - J_{03}) \end{aligned} \tag{86}$$

Each set of these generators satisfy the $\mathrm{SL}(2,\mathbb{R})$ algebra:

$$[J_1, J_2] = -J_0 \qquad\qquad [J_1, J_0] = J_2 \qquad\qquad [J_2, J_0] = J_1 \tag{87}$$

Since these isometries preserve the hyperboloid equation (81), they are also isometries of AdS_3. For later use it will prove convenient to use the generators:

$$J^3 = iJ_1 \qquad\qquad J^+ = -J_2 + iJ_3 \qquad\qquad J^- = J_2 + iJ_3 \tag{88}$$

which satisfy:

$$\left[J^0, J^\pm\right] = \pm J^\pm \qquad \left[J^+, J^-\right] = -2J^3 \tag{89}$$

The quadratic Casimir is then:

$$J^2 = \frac{1}{2}(J^+J^- + J^-J^+) - (J^3)^2 \tag{90}$$

In global coordinates these generators become

$$\begin{aligned} J^0 &= \frac{i}{2}\partial_u \\ J^+ &= \frac{i}{2}e^{-2iu}\left[\coth 2\rho\partial_u - \frac{1}{\sinh 2\rho}\partial_v + i\partial_\rho\right] \\ J^- &= \frac{i}{2}e^{2iu}\left[\coth 2\rho\partial_u - \frac{1}{\sinh 2\rho}\partial_v - i\partial_\rho\right] \end{aligned} \tag{91}$$

The generators of the other SL$(2, \mathbb{R})$ algebra are obtained by exchanging u and v in the above expressions.

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